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Appendices for

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Online Ad Procurement in Non-stationary Autobidding Worlds

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A Proofs for Section 4

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A.1 Additional definitions for Section 4

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Definition A.1 (Total variation between probability distributions). *Consider two distributions $\mathcal{P}, \mathcal{P}' \subseteq \Delta(\mathcal{S})$. Then we define their total variation as $\|\mathcal{P} - \mathcal{P}'\|_{TV} = \frac{1}{2} \int_{\mathcal{S}} |\mathcal{P}(s) - \mathcal{P}'(s)| ds$.*

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We also define the smoothed version of $h_t : \mathcal{X} \rightarrow \mathbb{R}$ (see Eq. (4)) for any t as followed:

$$\hat{h}_t(\mathbf{x}) = \mathbb{E}_{\mathbf{v} \sim U(\mathbb{B})} [\mathcal{L}_t(\mathbf{x} + \rho \mathbf{v}, \boldsymbol{\lambda}_t)] \quad (10)$$

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where we recall the Lagrangian function $\mathcal{L}_t$ is defined in Eq. (3).

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A.2 Additional lemmas for Section 4

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Lemma A.1 (Lipschitz continuity). *Let Assumption 2.1 hold, and recall the definitions $h_t(\mathbf{x})$ and $\hat{h}_t(\mathbf{x})$ from Eqs. (4) as well as (10), respectively, and recall $\boldsymbol{\lambda}_1 \dots \boldsymbol{\lambda}_T$ are the dual variables generated from Algorithm 1. Then for any $\mathbf{x}, \mathbf{x}' \in \mathcal{X}$, we have $|h_t(\mathbf{x}) - h_t(\mathbf{x}')| \leq (1 + K \frac{\bar{F}}{\beta}) L \cdot \|\mathbf{x} - \mathbf{x}'\|$ and*

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$$|h_t(\mathbf{x}) - \hat{h}_t(\mathbf{x})| \leq (1 + K \frac{\bar{F}}{\beta}) L \rho.$$

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Lemma A.2 (Bounding BOCO dynamic regret with surrogate loss). *Recall the definition $\hat{h}_t(\mathbf{x}) = \mathbb{E}_{\mathbf{v} \sim U(\mathbb{B})} [\mathcal{L}_t(\mathbf{x} + \rho \mathbf{v}, \boldsymbol{\lambda}_t)]$. Then, $\hat{h}_t(\mathbf{x})$ is concave. Further, For any $\mathbf{y} \in (1 - \alpha)\mathcal{X}$, we have $\hat{h}_t(\mathbf{y}) - \hat{h}_t(\tilde{\mathbf{x}}_t) \leq \mathbb{E}_{\mathbf{u}_t \sim U(\mathbb{S})} [\ell_t(\tilde{\mathbf{x}}_t) - \ell_t(\mathbf{y})]$, where $\tilde{\mathbf{x}}_t$ is defined in Eq. (5), and the surrogate loss function $\ell_t : \mathcal{X} \rightarrow \mathbb{R}$ is defined in Eq. (7).*

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Lemma A.3 (Bounding surrogate loss for each expert). *Recall the definition of individual forecasters $\tilde{\mathbf{x}}_t^i$ defined in Eq. (8), and the surrogate loss function $\ell_t : \mathcal{X} \rightarrow \mathbb{R}$ defined in Eq. (7). Then for any $i \in [N]$ and any sequence $\mathbf{y}_{1:T} \in \mathcal{X}^T$ we have (i) $\sum_{t \in [T]} \ell_t(\tilde{\mathbf{x}}_t^i) - \ell_t((1 - \alpha)\mathbf{y}_t) \leq \mathcal{O}\left(\frac{1 + P(\mathbf{y}_{1:T})}{\gamma_i} + \frac{\gamma_i}{\beta^2 \rho^2} T\right)$ and (ii) $\sum_{t \in [T]} \ell_t(\tilde{\mathbf{x}}_t) - \ell_t(\tilde{\mathbf{x}}_t^i) \leq \mathcal{O}(T\epsilon + \frac{1}{\epsilon})$. where the constant β is specified in Algorithm 1. Here, recall D is the diameter of the decision set $\mathcal{X}$.*

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The proofs of Lemmas A.1, A.2, A.3 are shown in Appendices A.9, A.10, and A.11, respectively.

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A.3 Proof for Lemma 4.1

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Proof. For any $k \in [K]$ we have

$$\begin{aligned} \sum_{t \in [T]} g_{k,t}(\mathbf{x}_t) &= \sum_{t \in [\tau_A - 1]} g_{k,t}(\mathbf{x}_t) + \sum_{t = \tau_A}^T g_{k,t}(\mathbf{x}_t) \stackrel{(a)}{\geq} \sum_{t \in [\tau_A - 1]} g_{k,t}(\mathbf{x}_t) + \bar{\beta}(T - \tau_A + 1) \\ &\geq \sum_{t \in [\tau_A - 1]} g_{k,t}(\mathbf{x}_t) + \beta(T - \tau_A) + \beta \stackrel{(b)}{\geq} \bar{G} + \beta > 0 \end{aligned} \quad (11)$$

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where in (a) we set $\mathbf{x}_t = \tilde{\mathbf{x}}_\beta$ for all $t = \tau_A \dots T$ and $g_{k,t}(\tilde{\mathbf{x}}_\beta) \geq \bar{\beta}$ for any $k \in [K]$; (b) follows from the definition of the stopping time such that for any $t' < \tau_A$ and $k \in [K]$ we have $\sum_{t \in [t']} g_{k,t}(\mathbf{x}_t) - \bar{G} + \beta(T - t' - 1) \geq 0$. $\square$

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A.4 Proof for Lemma 4.4

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Proof. It is easy to see $\boldsymbol{\lambda}_{t+1} = \Pi_{[0, \frac{F}{\beta} \mathbf{e}]}(\boldsymbol{\lambda}_t - \eta \nabla_{\boldsymbol{\lambda}} \mathcal{L}_t(\mathbf{x}_t, \boldsymbol{\lambda}_t))_+ = \arg \min_{\boldsymbol{\lambda} \in [0, \frac{F}{\beta} \mathbf{e}]} \nabla_{\boldsymbol{\lambda}} \mathcal{L}_t(\mathbf{x}_t, \boldsymbol{\lambda}_t)^\top \boldsymbol{\lambda} + \frac{1}{2\eta} \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_t\|^2$. By the first-order stationary condition

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526 at λ_{t+1} , we have for any $\lambda \in [0, \frac{\bar{F}}{\beta} e]$

$$\left(\nabla_{\lambda} \mathcal{L}_t(\mathbf{x}_t, \lambda_t) + \frac{1}{\eta} (\lambda_{t+1} - \lambda_t) \right)^{\top} (\lambda - \lambda_{t+1}) \geq 0$$

527 Then for all $\lambda \in \mathbb{R}_{\geq 0}^K$, it follows that

$$\begin{aligned} & \nabla_{\lambda} \mathcal{L}_t(\mathbf{x}_t, \lambda_t)^{\top} (\lambda_t - \lambda) \\ &= \nabla_{\lambda} \mathcal{L}_t(\mathbf{x}_t, \lambda_t)^{\top} (\lambda_t - \lambda_{t+1}) + \nabla_{\lambda} \mathcal{L}_t(\mathbf{x}_t, \lambda_t)^{\top} (\lambda_{t+1} - \lambda) \\ &\leq \nabla_{\lambda} \mathcal{L}_t(\mathbf{x}_t, \lambda_t)^{\top} (\lambda_t - \lambda_{t+1}) + \frac{1}{\eta} (\lambda_{t+1} - \lambda_t)^{\top} (\lambda - \lambda_{t+1}) \\ &\leq \nabla_{\lambda} \mathcal{L}_t(\mathbf{x}_t, \lambda_t)^{\top} (\lambda_t - \lambda_{t+1}) + \frac{1}{2\eta} \|\lambda - \lambda_t\|^2 - \frac{1}{2\eta} \|\lambda - \lambda_{t+1}\|^2 - \frac{1}{2\eta} \|\lambda_{t+1} - \lambda_t\|^2 \\ &\leq \frac{\eta}{2} \|\nabla_{\lambda} \mathcal{L}_t(\mathbf{x}_t, \lambda_t)\|^2 + \frac{1}{2\eta} \|\lambda - \lambda_t\|^2 - \frac{1}{2\eta} \|\lambda - \lambda_{t+1}\|^2 \end{aligned}$$

528 By a telescoping argument, we have

$$\begin{aligned} \sum_{\tau \in [t]} \nabla_{\lambda} \mathcal{L}_{\tau}(\mathbf{x}_{\tau}, \lambda_{\tau})^{\top} (\lambda_{\tau} - \lambda) &\leq \frac{\eta}{2} \sum_{\tau \in [t]} \|\nabla_{\lambda} \mathcal{L}_{\tau}(\mathbf{x}_{\tau}, \lambda_{\tau})\|^2 + \frac{1}{2\eta} \|\lambda - \lambda_1\|^2 \\ &= \frac{\eta}{2} \sum_{\tau \in [t]} \|\nabla_{\lambda} \mathcal{L}_{\tau}(\mathbf{x}_{\tau}, \lambda_{\tau})\|^2 + \frac{1}{2\eta} \|\lambda\|^2. \end{aligned} \quad (12)$$

529 where in the final equality we used $\lambda_1 = \mathbf{0}$. Also,

$$\|\nabla_{\lambda} \mathcal{L}_{\tau}(\mathbf{x}_{\tau}, \lambda_{\tau})\|^2 = \|\mathbf{g}_{\tau}(\mathbf{x}_{\tau})\|^2 \leq K \bar{G}^2 \quad (13)$$

530 Hence, combining Eqs. (12) and (13), we get the desired bound. $\square$

531 A.5 Proof of Lemma 4.5

532 *Proof.* If $\tau_A = T$, taking $\lambda = 0$ in Lemma 4.4 yields $\sum_{t \in [T]} \lambda_t^{\top} \mathbf{g}_t(\mathbf{x}_t) \leq \frac{\eta}{2} TK \bar{G}^2$ and thus the
533 desired inequality holds. If $\tau_A < T$, then there exists some $k \in [K]$ such that $\sum_{t \in [\tau_A]} g_{k,t}(\mathbf{x}_t) -$
534 $\bar{G} + \beta(T - \tau_A - 1) < 0$, so by taking $\lambda = \frac{\bar{F}}{\beta} e_k$ ($e_k \in \mathbb{R}^K$ is the unit vector whose k th entry is 1)
535 in Lemma 4.4 yields

$$\begin{aligned} & \sum_{t \in [\tau_A]} \lambda_t^{\top} \mathbf{g}_t(\mathbf{x}_t) \\ &\leq \sum_{t \in [\tau_A]} \lambda^{\top} \mathbf{g}_t(\mathbf{x}_t) + \frac{\eta}{2} TK \bar{G}^2 + \frac{1}{2\eta} \|\lambda\|^2 \\ &= \frac{\bar{F}}{\beta} \sum_{t \in [\tau_A]} g_{k,t}(\mathbf{x}_t) + \frac{\eta}{2} TK \bar{G}^2 + \frac{1}{2\eta} \left(\frac{\bar{F}}{\beta} \right)^2 \\ &\leq -\frac{\bar{F}}{\beta} \cdot \beta(T - \tau_A - 1) + \frac{\bar{F}}{\beta} \bar{G} + \frac{\eta}{2} TK \bar{G}^2 + \frac{1}{2\eta} \left(\frac{\bar{F}}{\beta} \right)^2 \\ &= -\bar{F}(T - \tau_A) + \bar{F} + \frac{\bar{F}}{\beta} \bar{G} + \frac{\eta}{2} TK \bar{G}^2 + \frac{1}{2\eta} \left(\frac{\bar{F}}{\beta} \right)^2 \end{aligned}$$

536 Summing with $\bar{F}(T - \tau_A)$ yields the desired result. $\square$

537 A.6 Proof of Lemma 4.6

538 *Proof.* Recall the definition of $\hat{h}_t(\mathbf{x})$ in Eq. (4). Then, we have

$$\begin{aligned} & \sum_{\tau \in [t]} h_{\tau}(\mathbf{y}_{\tau}) - \sum_{\tau \in [t]} h_{\tau}(\mathbf{x}_{\tau}) \\ &= \sum_{\tau \in [t]} \left(\underbrace{h_{\tau}(\mathbf{y}_{\tau}) - \hat{h}_{\tau}((1 - \alpha)\mathbf{y}_{\tau})}_A + \underbrace{\hat{h}_{\tau}((1 - \alpha)\mathbf{y}_{\tau}) - \hat{h}_{\tau}(\tilde{\mathbf{x}}_{\tau})}_B + \underbrace{\hat{h}_{\tau}(\tilde{\mathbf{x}}_{\tau}) - h_{\tau}(\mathbf{x}_{\tau})}_C \right) \end{aligned} \quad (14)$$

539 **Bounding A.**

$$\begin{aligned}
h_\tau(\mathbf{y}_\tau) - \hat{h}_\tau((1-\alpha)\mathbf{y}_\tau) &= h_\tau(\mathbf{y}_\tau) - h_\tau((1-\alpha)\mathbf{y}_\tau) + h_\tau((1-\alpha)\mathbf{y}_\tau) - \hat{h}_\tau((1-\alpha)\mathbf{y}_\tau) \\
&\stackrel{(a)}{\leq} (1 + K\frac{\bar{F}}{\beta})L\alpha\|\mathbf{y}_\tau\| + (1 + K\frac{\bar{F}}{\beta})L\rho \\
&\stackrel{(b)}{\leq} (1 + K\frac{\bar{F}}{\beta})L\alpha D + (1 + K\frac{\bar{F}}{\beta})L\rho
\end{aligned} \tag{15}$$

540 where (a) follows from Lemma A.1; (b) follows from $\|\mathbf{y}_\tau\| = \|\mathbf{y}_\tau - \mathbf{0}\| \leq D$ since we assumed
541 $\mathbf{0} \in \mathcal{X}$.

542 **Bounding B.**

$$\begin{aligned}
&\sum_{t \in [T]} \hat{h}_\tau((1-\alpha)\mathbf{y}_\tau) - \hat{h}_\tau(\tilde{\mathbf{x}}_\tau) \\
&\stackrel{(a)}{\leq} \sum_{t \in [T]} \mathbb{E}_{\mathbf{u}_\tau \sim U(\mathbb{S})} [\ell_\tau(\tilde{\mathbf{x}}_\tau) - \ell_\tau((1-\alpha)\mathbf{y}_\tau)] \\
&= \sum_{t \in [T]} \mathbb{E}_{\mathbf{u}_\tau \sim U(\mathbb{S})} [\ell_\tau(\tilde{\mathbf{x}}_\tau) - \ell_\tau(\tilde{\mathbf{x}}_\tau^i) + \ell_\tau(\tilde{\mathbf{x}}_\tau^i) - \ell_\tau((1-\alpha)\mathbf{y}_\tau)] \\
&\stackrel{(b)}{\leq} \mathcal{O}\left(\frac{P(\mathbf{y}_{1:T})}{\gamma_i} + \frac{\gamma_i K \frac{\bar{F}}{\beta} T}{\rho^2} + T\epsilon + \frac{1}{\epsilon}\right)
\end{aligned} \tag{16}$$

543 where (a) follows from Lemma A.2 and (b) follows from Lemma A.3 (i) and (ii).

544 **Bounding C.**

$$\begin{aligned}
\hat{h}_\tau(\tilde{\mathbf{x}}_\tau) - h_\tau(\mathbf{x}_\tau) &= \hat{h}_\tau(\tilde{\mathbf{x}}_\tau) - h_\tau(\tilde{\mathbf{x}}_\tau) + h_\tau(\tilde{\mathbf{x}}_\tau) - h_\tau(\mathbf{x}_\tau) \\
&\stackrel{(a)}{\leq} (1 + K\frac{\bar{F}}{\beta})L\rho + (1 + K\frac{\bar{F}}{\beta})L \cdot \|\tilde{\mathbf{x}}_\tau - \mathbf{x}_\tau\| \\
&\stackrel{(b)}{=} (1 + K\frac{\bar{F}}{\beta})L\rho + (1 + K\frac{\bar{F}}{\beta})L \cdot \|\rho\mathbf{u}_\tau\| \\
&\leq 2\rho(1 + K\frac{\bar{F}}{\beta})L
\end{aligned} \tag{17}$$

545 where (a) follows from Lemma A.1; (b) follows from the definition $\mathbf{x}_\tau = \tilde{\mathbf{x}}_\tau + \rho\mathbf{u}_\tau$ in Algorithm
546 1. $\square$

547 **A.7 Proof of Lemma 4.3**

548 **Stochastic.**

549 *Proof.* In the stochastic regime, we have $\mathcal{P} = \mathcal{P}_1 = \dots = \mathcal{P}_T$ for some $\mathcal{P}$, and therefore we can
550 rewrite $\text{OPT}(\mathcal{P}_{1:T})$ in Eq. (1) as followed

$$\text{OPT}(\mathcal{P}_{1:T}) = \max_{\mathbf{x}_{1:T} \in \mathcal{X}^T} \sum_{t \in [T]} F(\mathbf{x}_t) \quad \text{s.t.} \quad \sum_{t \in [T]} \mathbf{G}(\mathbf{x}_t) \geq \mathbf{0}.$$

551 where we defined $F(\mathbf{x}) = \mathbb{E}_{(f,g) \sim \mathcal{P}}[f(\mathbf{x})]$, and $\mathbf{G}(\mathbf{x}) = \mathbb{E}_{(f,g) \sim \mathcal{P}}[g(\mathbf{x})]$ for any $\mathbf{x} \in \mathcal{X}$. Hence, for
552 any $\boldsymbol{\lambda} \geq \mathbf{0}$ we have

$$\begin{aligned}
\text{OPT}(\mathcal{P}_{1:T}) &= \frac{T - \tau_A}{T} \text{OPT}(\mathcal{P}_{1:T}) + \frac{\tau_A}{T} \text{OPT}(\mathcal{P}_{1:T}) \\
&\leq (T - \tau_A)\bar{F} + \frac{\tau_A}{T} \max_{\mathbf{x}_{1:T} \in \mathcal{X}^T} \sum_{t \in [T]} (F(\mathbf{x}_t) + \boldsymbol{\lambda}^\top \mathbf{G}(\mathbf{x}_t)) \\
&= (T - \tau_A)\bar{F} + \frac{\tau_A}{T} \max_{\mathbf{x} \in \mathcal{X}} \sum_{t \in [T]} (F(\mathbf{x}) + \boldsymbol{\lambda}^\top \mathbf{G}(\mathbf{x})) \\
&= (T - \tau_A)\bar{F} + \tau_A \max_{\mathbf{x} \in \mathcal{X}} (F(\mathbf{x}) + \boldsymbol{\lambda}^\top \mathbf{G}(\mathbf{x}))
\end{aligned} \tag{18}$$

553 where in the inequality we applied Assumption 2.1 which states $\max_{\mathbf{x} \in \mathcal{X}} f(\mathbf{x})$ for all $(f, \mathbf{g}) \in \mathcal{S}$.
 554 Choosing $\boldsymbol{\lambda} = \boldsymbol{\lambda}_{\tau_A} := \frac{1}{\tau_A} \sum_{t \in [\tau_A]} \boldsymbol{\lambda}_t$ we have

$$\begin{aligned}
 \text{OPT}(\mathcal{P}_{1:T}) &\leq \mathbb{E} \left[(T - \tau_A) \bar{F} + \tau_A \max_{\mathbf{x} \in \mathcal{X}} (F(\mathbf{x}) + \boldsymbol{\lambda}^\top \mathbf{G}(\mathbf{x})) \right] \\
 &\leq \mathbb{E} \left[(T - \tau_A) \bar{F} + \max_{\mathbf{x} \in \mathcal{X}} \sum_{t \in [\tau_A]} (F(\mathbf{x}) + \boldsymbol{\lambda}_t^\top \mathbf{G}(\mathbf{x})) \right] \\
 &\stackrel{(a)}{\leq} \mathbb{E} \left[(T - \tau_A) \bar{F} + \max_{\mathbf{x} \in \mathcal{X}} \sum_{t \in [\tau_A]} \mathbb{E} \left[f_t(\mathbf{x}) + \boldsymbol{\lambda}_t^\top \mathbf{g}_t(\mathbf{x}) \mid \sigma(\mathcal{H}_{t-1}) \right] \right] \\
 &\stackrel{(b)}{=} \mathbb{E} \left[(T - \tau_A) \bar{F} + \max_{\mathbf{x} \in \mathcal{X}} \sum_{t \in [\tau_A]} \mathbb{E} \left[h_t(\mathbf{x}) \mid \sigma(\mathcal{H}_{t-1}) \right] \right] \\
 &\leq \mathbb{E} \left[(T - \tau_A) \bar{F} + \max_{\mathbf{x} \in \mathcal{X}} \sum_{t \in [\tau_A]} h_t(\mathbf{x}) \right]
 \end{aligned} \tag{19}$$

555 where in (a) we used the fact that $\boldsymbol{\lambda}_t$ is $\mathcal{H}_{t-1}$ -measurable; in (b) we used definitions $h_t(\mathbf{x}) =$
 556 $\mathcal{L}_t(\mathbf{x}; \boldsymbol{\lambda}_t)$ and $\mathcal{L}_t(\mathbf{x}; \boldsymbol{\lambda}) = f_t(\mathbf{x}) + \boldsymbol{\lambda}^\top \mathbf{g}_t(\mathbf{x})$ in Eqs. (3) and (4) respectively.

557 On the other hand, we have

$$f_t(\mathbf{x}_t) = h_t(\mathbf{x}_t) - \boldsymbol{\lambda}_t^\top \mathbf{g}_t(\mathbf{x}_t), \tag{20}$$

558 so combining this with Eq. (19) we have

$$\text{OPT}(\mathcal{P}_{1:T}) - \sum_{t \in [T]} \mathbb{E}[f_t(\mathbf{x}_t)] \leq \mathbb{E} \left[(T - \tau_A) \bar{F} + \max_{\mathbf{x} \in \mathcal{X}} \sum_{t \in [\tau_A]} (h_t(\mathbf{x}) - h_t(\mathbf{x}_t)) + \sum_{t \in \tau_A} \boldsymbol{\lambda}_t^\top \mathbf{g}_t(\mathbf{x}_t) \right] \tag{21}$$

559 where we also used the fact that $f_t(\mathbf{x}) \geq 0$ for all $t = \tau_A + 1 \dots T$ and $\mathbf{x} \in \mathcal{X}$. $\square$

560 **Adversarial.**

561 *Proof.* Recall the definition of ξ is Theorem 4.2:

$$\xi = 1 - \frac{\min_{(f, \mathbf{g}) \in \mathcal{S}} \min_{k \in [K], \mathbf{x} \in \mathcal{X}} g_k(\mathbf{x})}{\bar{\beta}} > 1 \tag{22}$$

562 For any $t \in [T]$, define $\tilde{\mathbf{y}}_t = \arg \max_{\mathbf{x}} f_t(\mathbf{x}) + \boldsymbol{\lambda}_t^\top \mathbf{g}_t(\mathbf{x})$.

563 By comparing to the safety action $\mathbf{x}_\beta \in \mathcal{X}$ which ensures $g_k(\mathbf{x}_\beta) \geq \bar{\beta}$ for any $k \in [K]$ and
 564 $(f, \mathbf{g}) \in \mathcal{S}$, as well as the optimal hindsight action $\mathbf{x}_t^* \in \mathcal{X}$ (i.e. $\mathbf{x}_1^* \dots \mathbf{x}_T^*$ is the optimal decision
 565 sequence to $\text{OPT}(\mathcal{P}_{1:T})$), we have

$$\begin{aligned}
 f_t(\tilde{\mathbf{y}}_t) + \boldsymbol{\lambda}_t^\top \mathbf{g}_t(\tilde{\mathbf{y}}_t) &\geq f_t(\mathbf{x}_\beta) + \boldsymbol{\lambda}_t^\top \mathbf{g}_t(\mathbf{x}_\beta) \geq \bar{\beta} \boldsymbol{\lambda}_t^\top \mathbf{e} \\
 f_t(\tilde{\mathbf{y}}_t) + \boldsymbol{\lambda}_t^\top \mathbf{g}_t(\tilde{\mathbf{y}}_t) &\geq f_t(\mathbf{x}_t^*) + \boldsymbol{\lambda}_t^\top \mathbf{g}_t(\mathbf{x}_t^*).
 \end{aligned} \tag{23}$$

566 We further have

$$\begin{aligned}
 \xi f_t(\tilde{\mathbf{y}}_t) &= f_t(\tilde{\mathbf{y}}_t) + (\xi - 1) f_t(\tilde{\mathbf{y}}_t) \\
 &\stackrel{(a)}{\geq} f_t(\mathbf{x}_t^*) + \boldsymbol{\lambda}_t^\top \mathbf{g}_t(\mathbf{x}_t^*) - \boldsymbol{\lambda}_t^\top \mathbf{g}_t(\tilde{\mathbf{y}}_t) + (\xi - 1) (-\boldsymbol{\lambda}_t^\top \mathbf{g}_t(\tilde{\mathbf{y}}_t) + \bar{\beta} \boldsymbol{\lambda}_t^\top \mathbf{e}) \\
 &= f_t(\mathbf{x}_t^*) + \boldsymbol{\lambda}_t^\top \mathbf{g}_t(\mathbf{x}_t^*) - \xi \boldsymbol{\lambda}_t^\top \mathbf{g}_t(\tilde{\mathbf{y}}_t) + (\xi - 1) \bar{\beta} \boldsymbol{\lambda}_t^\top \mathbf{e} \\
 &\stackrel{(b)}{\geq} f_t(\mathbf{x}_t^*) - \xi \boldsymbol{\lambda}_t^\top \mathbf{g}_t(\tilde{\mathbf{y}}_t)
 \end{aligned} \tag{24}$$

567 where (a) follows Eq.(23); in (b) we used the fact that $g_{k,t}(\mathbf{x}_t^*) + (\xi - 1)\bar{\beta} \geq 0$ since we have
 568 $\min_{(f,g) \in \mathcal{S}} \min_{k \in [K], \mathbf{x} \in \mathcal{X}} (g_{k,t}(\mathbf{x}) + (\xi - 1)\bar{\beta}) \geq 0$ (see Eq. (22)). Hence we have

$$\begin{aligned}
 & \text{OPT}(\mathcal{P}_{1:T}) - \sum_{t \in [T]} \mathbb{E}[f_t(\mathbf{x}_t)] \\
 &= \left(1 - \frac{1}{\xi}\right) \text{OPT}(\mathcal{P}_{1:T}) + \sum_{t \in [T]} \mathbb{E}\left[\frac{1}{\xi} f_t(\mathbf{x}_t^*) - f_t(\mathbf{x}_t)\right] \\
 &\leq \left(1 - \frac{1}{\xi}\right) \text{OPT}(\mathcal{P}_{1:T}) + \sum_{t \in [T]} \mathbb{E}\left[f_t(\tilde{\mathbf{y}}_t) - f_t(\mathbf{x}_t) + \boldsymbol{\lambda}_t^\top \mathbf{g}_t(\tilde{\mathbf{y}}_t)\right] \\
 &\leq \left(1 - \frac{1}{\xi}\right) \text{OPT}(\mathcal{P}_{1:T}) + \mathbb{E}\left[(T - \tau_A) \bar{F} + \sum_{t \in \tau_A} \left(f_t(\tilde{\mathbf{y}}_t) - f_t(\mathbf{x}_t) + \boldsymbol{\lambda}_t^\top \mathbf{g}_t(\tilde{\mathbf{y}}_t)\right)\right]
 \end{aligned} \tag{25}$$

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□

570 **δ -corrupted.**

571 Here, we will prove a more general δ -corrupted model where the input distribution sequence $\mathcal{P}_{1:T}$
 572 satisfies the following:

$$\sum_{t \in [T]} \|\mathcal{P}_t - \frac{1}{T} \sum_{s \in [T]} \mathcal{P}_s\|_{TV} \leq \delta \tag{26}$$

573 where the total variation norm is defined in Definition A.1. In fact, the definition in Section 2.2 for
 574 the δ -corrupted regime satisfies the above property: recall in the definition of Section 2.2, there exists
 575 $\mathcal{P} \in \Delta(\mathcal{S})$ as well as $\delta \in \mathbb{N}$ periods $\mathcal{T} = \{\tau_1 \dots \tau_\delta\} \subset [T]$ such that $\mathcal{P}_t = \mathcal{P}$ for all $t \notin \mathcal{T}$, hence
 576 for any $t \notin \mathcal{T}$, we have

$$\begin{aligned}
 \|\mathcal{P}_t - \frac{1}{T} \sum_{s \in [T]} \mathcal{P}_s\|_{TV} &= \|\mathcal{P} - \frac{1}{T} \left(T\mathcal{P} + \sum_{s \in \mathcal{T}} (\mathcal{P} - \mathcal{P}_s)\right)\|_{TV} \\
 &= \|\frac{1}{T} \sum_{s \in \mathcal{T}} (\mathcal{P} - \mathcal{P}_s)\|_{TV} \\
 &\leq \frac{\delta}{2T}
 \end{aligned} \tag{27}$$

577 On the other hand, we have for any $\tau \in \mathcal{T}$, $\|\mathcal{P}_\tau - \frac{1}{T} \sum_{s \in [T]} \mathcal{P}_s\|_{TV} \leq \frac{1}{2}$. Hence, summing up we
 578 get

$$\begin{aligned}
 \sum_{t \in [T]} \|\mathcal{P}_t - \frac{1}{T} \sum_{s \in [T]} \mathcal{P}_s\|_{TV} &= \sum_{t \in \mathcal{T}} \|\mathcal{P}_t - \frac{1}{T} \sum_{s \in [T]} \mathcal{P}_s\|_{TV} + \sum_{t \notin \mathcal{T}} \|\mathcal{P}_t - \frac{1}{T} \sum_{s \in [T]} \mathcal{P}_s\|_{TV} \\
 &\leq \frac{\delta}{2} + (T - \delta) \frac{\delta}{2T} \leq \delta
 \end{aligned}$$

579 which coincides with our general definition of δ -corruption in Eq. (26).

580 We now prove the δ -corruption regime under the general definition in Eq. (26). Define $\tilde{\mathcal{P}} =$
 581 $\frac{1}{T} \sum_{s \in [T]} \mathcal{P}_s$, $\tilde{F}(\mathbf{x}) = \mathbb{E}_{(f,g) \sim \tilde{\mathcal{P}}} [f(\mathbf{x})]$, $\tilde{\mathbf{G}}(\mathbf{x}) = \mathbb{E}_{(f,g) \sim \tilde{\mathcal{P}}} [g(\mathbf{x})]$, $F_t(\mathbf{x}) = \mathbb{E}_{(f,g) \sim \mathcal{P}_t} [f(\mathbf{x})]$ and
 582 $\mathbf{G}_t(\mathbf{x}) = \mathbb{E}_{(f,g) \sim \mathcal{P}_t} [g(\mathbf{x})]$ for all $t \in [T]$ and any $\mathbf{x} \in \mathcal{X}$. Then for any $\boldsymbol{\lambda} \in [\mathbf{0}, \frac{\bar{F}}{\bar{\beta}} \mathbf{e}]$, we have

$$\begin{aligned}
 \text{OPT}(\mathcal{P}_{1:T}) &\leq \max_{\mathbf{x}_{1:T} \in \mathcal{X}^T} \sum_{t \in [T]} (F_t(\mathbf{x}_t) + \boldsymbol{\lambda}^\top \mathbf{G}_t(\mathbf{x}_t)) \\
 &\leq \max_{\mathbf{x}_{1:T} \in \mathcal{X}} \sum_{t \in [T]} (\tilde{F}(\mathbf{x}_t) + \boldsymbol{\lambda}^\top \tilde{\mathbf{G}}(\mathbf{x}_t)) + (\bar{F} + \bar{G}K \frac{\bar{F}}{\bar{\beta}}) \delta \\
 &= T \cdot \max_{\mathbf{x} \in \mathcal{X}} (\tilde{F}(\mathbf{x}) + \boldsymbol{\lambda}^\top \tilde{\mathbf{G}}(\mathbf{x})) + (\bar{F} + \bar{G}K \frac{\bar{F}}{\bar{\beta}}) \delta,
 \end{aligned} \tag{28}$$

where the last inequality follows the definitions of $(\tilde{F}, \tilde{G})$, Assumption 2.1, and the general definition of δ -corruption in Eq. (26). After choosing $\bar{\lambda} = \frac{1}{\tau_A} \sum_{t \in [\tau_A]} \lambda_t$, similar to our proof in Eq. (19) for the stochastic case we have

$$\begin{aligned}
& \text{OPT}(\mathcal{P}_{1:T}) \\
&= \mathbb{E} \left[\frac{T - \tau_A}{T} \text{OPT}(\mathcal{P}_{1:T}) + \frac{\tau_A}{T} \text{OPT}(\mathcal{P}_{1:T}) \right] \\
&\stackrel{(a)}{\leq} \mathbb{E} \left[(T - \tau_A) \bar{F} + \tau_A \cdot \max_{\mathbf{x} \in \mathcal{X}} (\tilde{F}(\mathbf{x}) + \bar{\lambda}^\top \tilde{G}(\mathbf{x})) + \frac{\tau_A}{T} (\bar{F} + \bar{G} K \frac{\bar{F}}{\beta}) \delta \right] \\
&= \mathbb{E} \left[(T - \tau_A) \bar{F} + \frac{\tau_A}{T} (\bar{F} + \bar{G} K \frac{\bar{F}}{\beta}) \delta + \max_{\mathbf{x} \in \mathcal{X}} \left(\sum_{t \in [\tau_A]} \tilde{F}(\mathbf{x}) + \lambda_t^\top \tilde{G}(\mathbf{x}) \right) \right] \\
&\stackrel{(b)}{\leq} \mathbb{E} \left[(T - \tau_A) \bar{F} + \left(1 + \frac{\tau_A}{T} \right) (\bar{F} + \bar{G} K \frac{\bar{F}}{\beta}) \delta + \max_{\mathbf{x} \in \mathcal{X}} \sum_{t \in [\tau_A]} (F_t(\mathbf{x}) + \lambda_t^\top G_t(\mathbf{x})) \right] \\
&\leq \mathbb{E} \left[(T - \tau_A) \bar{F} + \left(1 + \frac{\tau_A}{T} \right) (\bar{F} + \bar{G} K \frac{\bar{F}}{\beta}) \delta + \max_{\mathbf{x} \in \mathcal{X}} \sum_{t \in [\tau_A]} \mathbb{E} \left[f_t(\mathbf{x}) + \lambda_t^\top g_t(\mathbf{x}) \mid \sigma(\mathcal{H}_{t-1}) \right] \right] \\
&\leq \mathbb{E} \left[(T - \tau_A) \bar{F} + 2(\bar{F} + \bar{G} K \frac{\bar{F}}{\beta}) \delta + \max_{\mathbf{x} \in \mathcal{X}} \sum_{t \in [\tau_A]} h_t(\mathbf{x}) \right],
\end{aligned} \tag{29}$$

where (a) follows from Eq. (28); (b) follows from the definition of general δ -corruption in Eq. (26). Finally, we complete the proof by using the definition $f_t(\mathbf{x}_t) = h_t(\mathbf{x}_t) - \lambda_t^\top g_t(\mathbf{x}_t)$ and following the same argument as in Eq. (21) for the stochastic regime.

Periodic.

Recall in Section 2.2 that in the periodic regime, there exists cycle length $q \in \mathbb{N}$ such that $T = cq$ for some integer $c \geq 2$ with $\mathcal{P}_{1:T}$ as $\mathcal{P}_{1:q} = \mathcal{P}_{q+1:2q} = \dots = \mathcal{P}_{(c-1)q+1:T}$. For any $t \in [T]$, define $c_t \in [c]$ such that $(c_t - 1)q + 1 \leq t \leq c_t q$. After denoting $\tilde{\mathcal{P}} = \frac{1}{q} \sum_{t \in [q]} \mathcal{P}_t$, we define the mean deviation within a single cycle of length q as

$$MD(\mathcal{P}_{1:q}) = \sum_{1 \leq t \leq q} \|\mathcal{P}_t - \tilde{\mathcal{P}}\|_{TV} \quad \text{and} \quad \delta = c \cdot MD(\mathcal{P}_{1:q}). \tag{30}$$

We define $\tilde{F}(\mathbf{x}) = \mathbb{E}_{(f,g) \sim \tilde{\mathcal{P}}} [f(\mathbf{x})]$, $\tilde{G}(\mathbf{x}) = \mathbb{E}_{(f,g) \sim \tilde{\mathcal{P}}} [g(\mathbf{x})]$, $F_t(\mathbf{x}) = \mathbb{E}_{(f,g) \sim \mathcal{P}_t} [f(\mathbf{x})]$ and $G_t(\mathbf{x}) = \mathbb{E}_{(f,g) \sim \mathcal{P}_t} [g(\mathbf{x})]$ for all $t \in [T]$ and any $\mathbf{x} \in \mathcal{X}$. Then for any $\lambda \in [0, \frac{\bar{F}}{\beta} \mathbf{e}]$, we have

$$\begin{aligned}
\text{OPT}(\mathcal{P}_{1:T}) &\leq \max_{\mathbf{x}_{1:T} \in \mathcal{X}^T} \sum_{t \in [T]} (F_t(\mathbf{x}_t) + \lambda^\top G_t(\mathbf{x}_t)) \\
&= c \cdot \max_{\mathbf{x}_{1:q} \in \mathcal{X}^q} \sum_{t \in [q]} (F_t(\mathbf{x}_t) + \lambda^\top G_t(\mathbf{x}_t)) \\
&\leq cq \cdot \max_{\mathbf{x} \in \mathcal{X}} (\tilde{F}(\mathbf{x}) + \bar{\lambda}^\top \tilde{G}(\mathbf{x})) + (\bar{F} + \bar{G} K \frac{\bar{F}}{\beta}) c \cdot MD(\mathcal{P}_{1:q}) \\
&\leq cq \cdot \max_{\mathbf{x} \in \mathcal{X}} (\tilde{F}(\mathbf{x}) + \bar{\lambda}^\top \tilde{G}(\mathbf{x})) + (\bar{F} + \bar{G} K \frac{\bar{F}}{\beta}) \delta,
\end{aligned}$$

where the equality follows the nature of periodic setting and the last inequality follows the definitions of $(\tilde{F}, \tilde{G})$, Assumption 2.1, and (30). After choosing $\lambda = \sum_{\hat{c} \in [c_{\tau_A}-1]} \frac{q}{\tau_A} \lambda_{(\hat{c}-1)q+1} +$

598 $\frac{\tau_A - (c_{\tau_A} - 1)q}{\tau_A} \boldsymbol{\lambda}_{(c_{\tau_A} - 1)q+1}$, we further have that

$$\begin{aligned}
& \text{OPT}(\mathcal{P}_{1:T}) \\
&= \frac{T - \tau_A}{T} \text{OPT}(\mathcal{P}_{1:T}) + \frac{\tau_A}{T} \text{OPT}(\mathcal{P}_{1:T}) \\
&\leq (T - \tau_A) \bar{F} + \tau_A \cdot \max_{\mathbf{x} \in \mathcal{X}} (\tilde{F}(\mathbf{x}) + \boldsymbol{\lambda}^\top \tilde{\mathbf{G}}(\mathbf{x})) + \frac{\tau_A}{T} (\bar{F} + \bar{G}K \frac{\bar{F}}{\beta}) \delta \\
&= (T - \tau_A) \bar{F} + \max_{\mathbf{x} \in \mathcal{X}} \left(\tau_A \tilde{F}(\mathbf{x}) + \left(\sum_{\hat{c} \in [c_{\tau_A} - 1]} q \boldsymbol{\lambda}_{(\hat{c} - 1)q+1} + (\tau_A - (c_{\tau_A} - 1)q) \boldsymbol{\lambda}_{(c_{\tau_A} - 1)q+1} \right)^\top \tilde{\mathbf{G}}(\mathbf{x}) \right) \\
&\quad + \frac{\tau_A}{T} (\bar{F} + \bar{G}K \frac{\bar{F}}{\beta}) \delta \\
&= (T - \tau_A) \bar{F} + \max_{\mathbf{x} \in \mathcal{X}} \left(q \cdot \sum_{\hat{c} \in [c_{\tau_A} - 1]} \left(\tilde{F}(\mathbf{x}) + \boldsymbol{\lambda}_{(\hat{c} - 1)q+1}^\top \tilde{\mathbf{G}}(\mathbf{x}) \right) \right. \\
&\quad \left. + (\tau_A - (c_{\tau_A} - 1)q) \cdot \left(\tilde{F}(\mathbf{x}) + \boldsymbol{\lambda}_{(c_{\tau_A} - 1)q+1}^\top \tilde{\mathbf{G}}(\mathbf{x}) \right) \right) + \frac{\tau_A}{T} (\bar{F} + \bar{G}K \frac{\bar{F}}{\beta}) \delta \\
&\leq (T - \tau_A) \bar{F} + \max_{\mathbf{x} \in \mathcal{X}} \sum_{t \in [\tau_A]} \left(\tilde{F}(\mathbf{x}) + \boldsymbol{\lambda}_t^\top \tilde{\mathbf{G}}(\mathbf{x}) \right) + \bar{G} \cdot \sum_{t \in [\tau_A]} \|\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_{(c_t - 1)q+1}\|_1 + \frac{\tau_A}{T} (\bar{F} + \bar{G}K \frac{\bar{F}}{\beta}) \delta.
\end{aligned}$$

599 From (8) in Algorithm 1, we know that $\|\boldsymbol{\lambda}_{t+1} - \boldsymbol{\lambda}_t\|_1 \leq \eta \bar{G}K$, which further implies $\|\boldsymbol{\lambda}_{t+i} - \boldsymbol{\lambda}_t\|_1 \leq$
600 $\eta \bar{G}K i$ for any $i \in [q - 1]$ and thus

$$\sum_{t \in [\tau_A]} \|\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_{(c_t - 1)q+1}\|_1 \leq c_{\tau_A} \eta \bar{G}K \sum_{i \in [q-1]} i \leq \frac{1}{2} \bar{G}K \eta c_{\tau_A} q^2. \quad (31)$$

601 After combining the two equations above, it follows that

$$\begin{aligned}
& \text{OPT}(\mathcal{P}_{1:T}) \\
&\leq \mathbb{E} \left[(T - \tau_A) \bar{F} + \max_{\mathbf{x} \in \mathcal{X}} \sum_{t \in [\tau_A]} \left(\tilde{F}(\mathbf{x}) + \boldsymbol{\lambda}_t^\top \tilde{\mathbf{G}}(\mathbf{x}) \right) + \frac{1}{2} \bar{G}^2 K \eta c_{\tau_A} q^2 + \frac{\tau_A}{T} (\bar{F} + \bar{G}K \frac{\bar{F}}{\beta}) \delta \right] \\
&\leq \mathbb{E} \left[(T - \tau_A) \bar{F} + \max_{\mathbf{x} \in \mathcal{X}} \sum_{t \in [\tau_A]} (F_t(\mathbf{x}) + \boldsymbol{\lambda}_t^\top \mathbf{G}_t(\mathbf{x})) + \frac{1}{2} \bar{G}^2 K \eta c_{\tau_A} q^2 + 2(\bar{F} + \bar{G}K \frac{\bar{F}}{\beta}) \delta \right] \\
&\leq \mathbb{E} \left[(T - \tau_A) \bar{F} + 2(\bar{F} + \bar{G}K \frac{\bar{F}}{\beta}) \delta + \frac{1}{2} \bar{G}^2 K \eta q T + \max_{\mathbf{x} \in \mathcal{X}} \sum_{t \in [\tau_A]} \mathbb{E} \left[h_t(\mathbf{x}) \mid \sigma(\mathcal{H}_{t-1}) \right] \right] \\
&\leq \mathbb{E} \left[(T - \tau_A) \bar{F} + 2(\bar{F} + \bar{G}K \frac{\bar{F}}{\beta}) \delta + \frac{1}{2} \bar{G}^2 K \eta q T + \max_{\mathbf{x} \in \mathcal{X}} \sum_{t \in [\tau_A]} h_t(\mathbf{x}) \right]
\end{aligned}$$

602 where the second last inequality follows from $c_{\tau_A} q \leq cq = T$.

603 Finally, we complete the proof by using the definition $f_t(\mathbf{x}_t) = h_t(\mathbf{x}_t) - \boldsymbol{\lambda}_t^\top \mathbf{g}_t(\mathbf{x}_t)$ and following
604 the same argument as in Eq. (21) for the stochastic regime.

605 Ergodic.

606 Consider some $\kappa \geq \log(T)$. Given the input distribution sequence $\mathcal{P}_{1:T}$, denote $\mathcal{P}_{(t+\kappa)[t-1]}$ as the
607 conditional distribution of $(f_{t+\kappa}, \mathbf{g}_{t+\kappa})$ conditioned on the $\{(f_\tau, \mathbf{g}_\tau)\}_{\tau \in [t]}$. Then, in the ergodic
608 regime, there exists a stationary distribution $\tilde{\mathcal{P}} \in \Delta(\mathcal{S})$ and absolute constant $R > 0$ such that

$$\sup_{\{(f_t, \mathbf{g}_t)\}_{t \in [T]} \in \mathcal{S}^T} \sup_{t \in [T-\kappa]} \|\mathcal{P}_{(t+\kappa)[t-1]} - \tilde{\mathcal{P}}\|_{TV} \leq \delta := R \exp(-\kappa) \quad (32)$$

609 By defining $\tilde{F}(\mathbf{x}) = \mathbb{E}_{(f, \mathbf{g}) \sim \tilde{\mathcal{P}}} [f(\mathbf{x})]$, $\tilde{\mathbf{G}}(\mathbf{x}) = \mathbb{E}_{(f, \mathbf{g}) \sim \tilde{\mathcal{P}}} [\mathbf{g}(\mathbf{x})]$, $\hat{F}_{t+\kappa}(\mathbf{x}) =$
610 $\mathbb{E}_{(f, \mathbf{g}) \sim \mathcal{P}_{(t+\kappa)[t-1]}} [f(\mathbf{x})]$, $\hat{\mathbf{G}}_{t+\kappa}(\mathbf{x}) = \mathbb{E}_{(f, \mathbf{g}) \sim \mathcal{P}_{(t+\kappa)[t-1]}} [\mathbf{g}(\mathbf{x})]$, $F_t(\mathbf{x}) = \mathbb{E}_{(f, \mathbf{g}) \sim \mathcal{P}_t} [f(\mathbf{x})]$

611 and $\mathbf{G}_t(\mathbf{x}) = \mathbb{E}_{(f,g) \sim \mathcal{P}_t}[\mathbf{g}(\mathbf{x})]$ for all $t \in [T]$ and any $\mathbf{x} \in \mathcal{X}$, we know that for any $\boldsymbol{\lambda} \in [0, \frac{\bar{F}}{\beta} \mathbf{e}]$, it
 612 follows that

$$\begin{aligned}
 & \text{OPT}(\mathcal{P}_{1:T}) \\
 & \leq \max_{\mathbf{x}_{1:T} \in \mathcal{X}^T} \mathbb{E} \left[\sum_{t \in [T]} (F_t(\mathbf{x}_t) + \boldsymbol{\lambda}^\top \mathbf{G}_t(\mathbf{x}_t)) \right] \\
 & = \max_{\mathbf{x}_{1:\kappa} \in \mathcal{X}^\kappa} \mathbb{E} \left[\sum_{t \in [\kappa]} (F_t(\mathbf{x}_t) + \boldsymbol{\lambda}^\top \mathbf{G}_t(\mathbf{x}_t)) \right] + \max_{\mathbf{x}_{\kappa+1:T} \in \mathcal{X}^{T-\kappa}} \mathbb{E} \left[\sum_{t=1}^{T-\kappa} (\hat{F}_{t+\kappa}(\mathbf{x}_{t+\kappa}) + \boldsymbol{\lambda}^\top \hat{\mathbf{G}}_{t+\kappa}(\mathbf{x}_{t+\kappa})) \right] \\
 & \leq (\bar{F} + \bar{G}K \frac{\bar{F}}{\beta})\kappa + \max_{\mathbf{x}_{\kappa+1:T} \in \mathcal{X}^{T-\kappa}} \sum_{t=1}^{T-\kappa} (\tilde{F}(\mathbf{x}_{t+\kappa}) + \boldsymbol{\lambda}^\top \tilde{\mathbf{G}}(\mathbf{x}_{t+\kappa})) + (\bar{F} + \bar{G}K \frac{\bar{F}}{\beta}) \cdot (T - \kappa)\delta \\
 & \leq T \cdot \max_{\mathbf{x} \in \mathcal{X}} (\tilde{F}(\mathbf{x}) + \boldsymbol{\lambda}^\top \tilde{\mathbf{G}}(\mathbf{x})) + (\bar{F} + \bar{G}K \frac{\bar{F}}{\beta})\kappa + (\bar{F} + \bar{G}K \frac{\bar{F}}{\beta}) \cdot T\delta
 \end{aligned} \tag{33}$$

613 By choosing $\boldsymbol{\lambda} = \frac{1}{\tau_A} \sum_{t \in [\tau_A]} \boldsymbol{\lambda}_t$, we further have

$$\begin{aligned}
 & \text{OPT}(\mathcal{P}_{1:T}) \\
 & = \mathbb{E} \left[\frac{T - \tau_A}{T} \text{OPT}(\mathcal{P}_{1:T}) + \frac{\tau_A}{T} \text{OPT}(\mathcal{P}_{1:T}) \right] \\
 & \leq \mathbb{E} \left[(T - \tau_A) \bar{F} + \tau_A \cdot \max_{\mathbf{x} \in \mathcal{X}} (\tilde{F}(\mathbf{x}) + \boldsymbol{\lambda}^\top \tilde{\mathbf{G}}(\mathbf{x})) \right] + (\bar{F} + \bar{G}K \frac{\bar{F}}{\beta})\kappa + (\bar{F} + \bar{G}K \frac{\bar{F}}{\beta}) \cdot T\delta \\
 & = \mathbb{E} \left[(T - \tau_A) \bar{F} + \max_{\mathbf{x} \in \mathcal{X}} \sum_{t \in [\tau_A]} (\tilde{F}(\mathbf{x}) + \boldsymbol{\lambda}_t^\top \tilde{\mathbf{G}}(\mathbf{x})) \right] + (\bar{F} + \bar{G}K \frac{\bar{F}}{\beta})\kappa + (\bar{F} + \bar{G}K \frac{\bar{F}}{\beta}) \cdot T\delta \\
 & \leq \mathbb{E} \left[(T - \tau_A) \bar{F} + \max_{\mathbf{x} \in \mathcal{X}} \sum_{t \in [\tau_A]} (\hat{F}_{t+\kappa}(\mathbf{x}) + \boldsymbol{\lambda}_t^\top \hat{\mathbf{G}}_{t+\kappa}(\mathbf{x})) \right] + (\bar{F} + \bar{G}K \frac{\bar{F}}{\beta})\kappa + 2(\bar{F} + \bar{G}K \frac{\bar{F}}{\beta}) \cdot T\delta \\
 & = \mathbb{E} \left[(T - \tau_A) \bar{F} + \max_{\mathbf{x} \in \mathcal{X}} \sum_{t \in [\tau_A]} (\hat{F}_{t+\kappa}(\mathbf{x}) + \boldsymbol{\lambda}_{t+\kappa}^\top \hat{\mathbf{G}}_{t+\kappa}(\mathbf{x}) + (\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_{t+\kappa})^\top \hat{\mathbf{G}}_{t+\kappa}(\mathbf{x})) \right] \\
 & \quad + (\bar{F} + \bar{G}K \frac{\bar{F}}{\beta})\kappa + 2(\bar{F} + \bar{G}K \frac{\bar{F}}{\beta}) \cdot T\delta \\
 & \stackrel{(a)}{\leq} \mathbb{E} \left[(T - \tau_A) \bar{F} + \max_{\mathbf{x} \in \mathcal{X}} \sum_{t \in [\tau_A]} (\hat{F}_{t+\kappa}(\mathbf{x}) + \boldsymbol{\lambda}_{t+\kappa}^\top \hat{\mathbf{G}}_{t+\kappa}(\mathbf{x})) \right] + \kappa \eta T K \bar{G}^2 \\
 & \quad + (\bar{F} + \bar{G}K \frac{\bar{F}}{\beta})\kappa + 2(\bar{F} + \bar{G}K \frac{\bar{F}}{\beta}) \cdot T\delta \\
 & \leq \mathbb{E} \left[(T - \tau_A) \bar{F} + \max_{\mathbf{x} \in \mathcal{X}} \sum_{t \in [\tau_A - \kappa]} (\hat{F}_{t+\kappa}(\mathbf{x}) + \boldsymbol{\lambda}_{t+\kappa}^\top \hat{\mathbf{G}}_{t+\kappa}(\mathbf{x})) \right] + \kappa \eta T K \bar{G}^2 \\
 & \quad + 2(\bar{F} + \bar{G}K \frac{\bar{F}}{\beta})\kappa + 2(\bar{F} + \bar{G}K \frac{\bar{F}}{\beta}) \cdot T\delta \\
 & \stackrel{(b)}{\leq} \mathbb{E} \left[(T - \tau_A) \bar{F} + \max_{\mathbf{x} \in \mathcal{X}} \sum_{t=\kappa+1}^{\tau_A} h_t(\mathbf{x}) \right] + \kappa \eta T K \bar{G}^2 + 2(\bar{F} + \bar{G}K \frac{\bar{F}}{\beta})\kappa + 2(\bar{F} + \bar{G}K \frac{\bar{F}}{\beta}) \cdot T\delta \\
 & \leq \mathbb{E} \left[(T - \tau_A) \bar{F} + \max_{\mathbf{x} \in \mathcal{X}} \sum_{t \in [\tau_A]} h_t(\mathbf{x}) \right] + \kappa \eta T K \bar{G}^2 + 2(\bar{F} + \bar{G}K \frac{\bar{F}}{\beta})\kappa + 2(\bar{F} + \bar{G}K \frac{\bar{F}}{\beta}) \cdot T\delta \\
 & \stackrel{(c)}{\leq} \mathbb{E} \left[(T - \tau_A) \bar{F} + \max_{\mathbf{x} \in \mathcal{X}} \sum_{t \in [\tau_A]} h_t(\mathbf{x}) \right] + \kappa \eta T K \bar{G}^2 + 2(\bar{F} + \bar{G}K \frac{\bar{F}}{\beta})\kappa + 2R(\bar{F} + \bar{G}K \frac{\bar{F}}{\beta}),
 \end{aligned} \tag{34}$$

where in (a), from (8) in Algorithm 1, we know that $\|\lambda_{t+1} - \lambda_t\|_1 \leq \eta \bar{G} K$, which further implies $\|\lambda_{t+\kappa} - \lambda_t\|_1 \leq \kappa \eta \bar{G} K$ and thus

$$(\lambda_t - \lambda_{t+\kappa})^\top \hat{G}_{t+\kappa}(\mathbf{x}_t) \leq \kappa \eta K \bar{G}^2 \quad (35)$$

In (b), we used the fact that for any $t \geq \kappa + 1$, we have

$$\begin{aligned} & \mathbb{E} \left[\max_{x \in \mathcal{X}} \sum_{t \in [\tau_A - \kappa]} (\hat{F}_{t+\kappa}(\mathbf{x}) + \lambda_{t+\kappa}^\top \hat{G}_{t+\kappa}(\mathbf{x})) \right] \\ &= \mathbb{E} \left[\max_{x \in \mathcal{X}} \sum_{t \in [\tau_A - \kappa]} \mathbb{E} [h_{t+\kappa}(\mathbf{x}) \mid (f_\tau, \mathbf{g}_\tau)_{\tau \in [t-1]}] \right] \\ &\leq \mathbb{E} \left[\max_{x \in \mathcal{X}} \sum_{t \in [\tau_A - \kappa]} h_{t+\kappa}(\mathbf{x}) \right] \end{aligned} \quad (36)$$

In (c) we used the fact that $\kappa \geq \log(T)$, so $\delta = R \exp(-\kappa) \geq R$.

Finally, we complete the proof by using the definition $f_t(\mathbf{x}_t) = h_t(\mathbf{x}_t) - \lambda_t^\top \mathbf{g}_t(\mathbf{x}_t)$ and following the same argument as in Eq. (21) for the stochastic regime.

A.8 Proof of Theorem 4.2

Proof. We bound the regret in every world as followed

$$\begin{aligned} \mathcal{R}_T &= \text{OPT}(\mathcal{P}_{1:T}) - \sum_{t \in [T]} \mathbb{E} [f_t(\mathbf{x}_t)] \\ &\stackrel{(a)}{\leq} \mathbb{E} \left[\bar{F}(T - \tau_A) + \sum_{t \in [\tau_A]} \lambda_t^\top \mathbf{g}_t(\mathbf{x}_t) + \mathcal{R}_{\text{BOCO}}(\tau_A) \right] \\ &\stackrel{(b)}{\leq} \mathbb{E} [\mathcal{R}_{\text{BOCO}}(\tau_A)] \end{aligned}$$

where (a) follows from Lemma 4.3, and (b) follows from Lemma 4.5. Recall $\mathcal{R}_{\text{BOCO}}(\tau_A)$ is specified in Lemma 4.3 for each world.

In the following we bound $\mathcal{R}_{\text{BOCO}}(\tau_A)$ for each world.

Stochastic.

$$\mathbb{E} [\mathcal{R}_{\text{BOCO}}(\tau_A)] = \mathbb{E} \left[\max_{\mathbf{x} \in \mathcal{X}} \sum_{t \in [\tau_A]} h_t(\mathbf{x}) - h_t(\mathbf{x}_t) \right] \stackrel{(a)}{\leq} \mathcal{O} \left(\frac{\rho T}{\beta} + \frac{1}{\gamma_i} + \frac{\gamma_i K T}{\beta^2 \rho^2} + T\epsilon + \frac{1}{\epsilon} \right) \stackrel{(b)}{=} \mathcal{O} \left(T^{\frac{3}{4}} \right) \quad (37)$$

where (a) follows from Lemma 4.6 by taking the comparator sequence $\mathbf{y}_t = \arg \max_{\mathbf{x} \in \mathcal{X}} \sum_{t \in [\tau_A]} h_t(\mathbf{x})$ for all $t \in [\tau_A]$ such that $P(\mathbf{y}_{1:T}) = 1$, as well as any primal ascent expert $i \in [N]$; (b) follows from taking $\eta = \frac{1}{\sqrt{KT}}$, $\rho = K^{\frac{1}{3}} T^{-\frac{1}{4}}$, $\epsilon = T^{-\frac{1}{2}}$, $\beta = \frac{1}{\log(T)}$, and finally choosing $\gamma_i = K^{-\frac{1}{6}} (1 + DT)^{\frac{1}{2}} T^{-\frac{3}{4}}$. Recall all primal ascent expert stepsizes are $\{\gamma_1 \dots \gamma_N\} = \{2^{-i} K^{-\frac{1}{6}} (1 + DT)^{\frac{1}{2}} T^{-\frac{3}{4}} : i = 0 \dots N\}$.

δ -corrupted, Periodic, and Ergodic. The proof is nearly identical with that of the stochastic world in Eq. (37) given that we still consider the comparator sequence $\mathbf{y}_t = \arg \max_{\mathbf{x} \in \mathcal{X}} \sum_{t \in [\tau_A]} h_t(\mathbf{x})$ for all $t \in [\tau_A]$ such that $P(\mathbf{y}_{1:T}) = 1$. Hence we will omit the proof.

Adversarial. Recall the definition $\tilde{\mathbf{y}}_t = \arg \max_{\mathbf{x} \in \mathcal{X}} f_t(\mathbf{x}_t) + \lambda_t^\top \mathbf{g}(\mathbf{x}_t)$. Then we have

$$\begin{aligned} \mathbb{E} [\mathcal{R}_{\text{BOCO}}(\tau_A)] &= \left(1 - \frac{1}{\xi} \right) \text{OPT}(\mathcal{P}_{1:T}) + \sum_{t \in [\tau_A]} \mathbb{E} [h_t(\tilde{\mathbf{y}}_t) - h_t(\mathbf{x}_t)] \\ &\leq \mathcal{O} \left(\frac{\rho T}{\beta} + \frac{1 + P(\tilde{\mathbf{y}}_{1:T})}{\gamma_i} + \frac{\gamma_i K T}{\beta^2 \rho^2} + T\epsilon + \frac{1}{\epsilon} \right) = \left(1 - \frac{1}{\xi} \right) \text{OPT}(\mathcal{P}_{1:T}) + o(T) \end{aligned} \quad (38)$$

635 where we chose the primal ascent stepsize γ_i s.t.

$$\frac{1}{2}K^{-\frac{1}{6}}(1 + P(\tilde{\mathbf{y}}_{1:T}))^{\frac{1}{2}}T^{-\frac{3}{4}} \leq \gamma_i \leq K^{-\frac{1}{6}}(1 + P(\tilde{\mathbf{y}}_{1:T}))^{\frac{1}{2}}T^{-\frac{3}{4}} \quad (39)$$

636 We note that such a γ_i must exist because $P(\tilde{\mathbf{y}}_{1:T}) \leq DT$ given all $\tilde{\mathbf{y}}_t \in \mathcal{X}$, so that the largest
 637 element in the primal ascent stepsize set, namely $K^{-\frac{1}{6}}(1 + DT)^{\frac{1}{2}}T^{-\frac{3}{4}}$ is larger than the upper bound
 638 above, namely $K^{-\frac{1}{6}}(1 + P(\tilde{\mathbf{y}}_{1:T}))^{\frac{1}{2}}T^{-\frac{3}{4}}$.

639 $\square$

640 A.9 Proof for Lemma A.1

641 *Proof.* Recall the definition $h_t(\mathbf{x}) = f_t(\mathbf{x}) + \boldsymbol{\lambda}_t^\top \mathbf{g}_t(\mathbf{x})$ in Eq. (4). Then we have

$$\begin{aligned} \left| h_t(\mathbf{x}) - h_t(\mathbf{x}') \right| &\leq \left| f_t(\mathbf{x}) - f_t(\mathbf{x}') \right| + \|\boldsymbol{\lambda}_t\| \cdot \|\mathbf{g}_t(\mathbf{x}) - \mathbf{g}_t(\mathbf{x}')\| \\ &\stackrel{(a)}{\leq} L\|\mathbf{x} - \mathbf{x}'\| + K\frac{\bar{F}}{\beta}L\|\mathbf{x} - \mathbf{x}'\| = (1 + K\frac{\bar{F}}{\beta})L \cdot \|\mathbf{x} - \mathbf{x}'\| \end{aligned} \quad (40)$$

642 where (a) follows from the fact that any $(f, \mathbf{g}) \in \mathcal{S}$ are L -lipschitz under Assumption 2.1.

643 On the other hand, recall the definition $\hat{h}_t(\mathbf{x}) = \mathbb{E}_{\mathbf{v} \sim U(\mathbb{B})}[\mathcal{L}_t(\mathbf{x} + \rho\mathbf{v}, \boldsymbol{\lambda}_t)]$ in Eq. (10). Then we have

$$\left| h_t(\mathbf{x}) - \hat{h}_t(\mathbf{x}) \right| = \mathbb{E}_{\mathbf{v} \sim U(\mathbb{B})} \left[h_t(\mathbf{x}) - h_t(\mathbf{x} + \rho\mathbf{v}) \right] \leq (1 + K\frac{\bar{F}}{\beta})L\rho \cdot \mathbb{E}_{\mathbf{v} \sim U(\mathbb{B})}[\|\mathbf{v}\|] = (1 + K\frac{\bar{F}}{\beta})L\rho \quad (41)$$

644 where the inequality follows from the first part of this lemma. $\square$

645 A.10 Proof of Lemma A.2

646 *Proof.* Recall the definitions $h_t(\mathbf{x}) = f_t(\mathbf{x}) + \boldsymbol{\lambda}_t^\top \mathbf{g}_t(\mathbf{x})$ in Eq. (4), and $\hat{h}_t(\mathbf{x}) = \mathbb{E}_{\mathbf{v} \sim U(\mathbb{B})}[\mathcal{L}_t(\mathbf{x} +$
 647 $\rho\mathbf{v}, \boldsymbol{\lambda}_t)]$ in Eq. (10). Then, we have

$$\begin{aligned} \hat{h}_t(\mathbf{y}) - \hat{h}_t(\tilde{\mathbf{x}}_t) &\stackrel{(a)}{\leq} \langle \nabla \hat{h}_t(\tilde{\mathbf{x}}_t), \mathbf{y} - \tilde{\mathbf{x}}_t \rangle \\ &\stackrel{(b)}{=} \left\langle \frac{d}{\rho} \cdot \mathbb{E}_{\mathbf{u} \sim U(\mathbb{S})} [h_t(\tilde{\mathbf{x}}_t + \rho\mathbf{u}) \cdot \mathbf{u}] , \mathbf{y} - \tilde{\mathbf{x}}_t \right\rangle \\ &= \mathbb{E}_{\mathbf{u}_t \sim U(\mathbb{S})} \left[\left\langle \frac{d}{\rho} \cdot h_t(\tilde{\mathbf{x}}_t + \rho\mathbf{u}_t) \cdot \mathbf{u}_t , \mathbf{y} - \tilde{\mathbf{x}}_t \right\rangle \right] \\ &\stackrel{(c)}{=} \mathbb{E}_{\mathbf{u}_t \sim U(\mathbb{S})} [\langle \nabla_t, \mathbf{y} - \tilde{\mathbf{x}}_t \rangle] \\ &\stackrel{(d)}{=} \mathbb{E}_{\mathbf{u}_t \sim U(\mathbb{S})} [\ell_t(\tilde{\mathbf{x}}_t) - \ell_t(\mathbf{y})] \end{aligned} \quad (42)$$

648 where (a) follows from concavity of $\hat{h}_t(\cdot)$; (b) follows from Lemma B.2 by taking $h = -h_t$,
 649 so that in the lemma $-\nabla_{\mathbf{x}} \mathbb{E}_{\mathbf{v} \sim U(\mathbb{B})}[h(\mathbf{x} + \rho\mathbf{v})] = \nabla \hat{h}_t(\mathbf{x})$ and $-\mathbb{E}_{\mathbf{u} \sim U(\mathbb{S})}[h(\mathbf{x} + \rho\mathbf{u}) \cdot \mathbf{u}] =$
 650 $\mathbb{E}_{\mathbf{u} \sim U(\mathbb{S})}[h_t(\mathbf{x} + \rho\mathbf{u}) \cdot \mathbf{u}]$; (c) follows from the gradient estimate in Eq. (6) where

$$\nabla_t = \frac{d}{\rho} (f_t(\mathbf{x}_t) + \boldsymbol{\lambda}_t^\top \mathbf{g}_t(\mathbf{x}_t)) \cdot \mathbf{u}_t = \frac{d}{\rho} \cdot h_t(\mathbf{x}_t) \cdot \mathbf{u}_t = \frac{d}{\rho} \cdot h_t(\tilde{\mathbf{x}}_t + \rho\mathbf{u}_t) \cdot \mathbf{u}_t$$

651 Finally, (d) follows from the definition of surrogate loss functions in Eq. (7). $\square$

652 A.11 Proof of Lemma A.3

653 **Proving (i):**

654 *Proof.* Since $\tilde{\mathbf{x}}_{t+1}^i = \Pi_{(1-\alpha)\mathcal{X}}(\tilde{\mathbf{x}}_t^i + \gamma_i \nabla_t)$ we have $\|\mathbf{y} - \tilde{\mathbf{x}}_{t+1}^i\| \leq \|\mathbf{y} - (\tilde{\mathbf{x}}_t^i + \gamma_i \nabla_t)\|$ for any
 655 $\mathbf{y} \in (1 - \alpha)\mathcal{X}$. Then

$$\begin{aligned} \|\mathbf{y} - \tilde{\mathbf{x}}_{t+1}^i\|^2 &\leq \|\mathbf{y} - \tilde{\mathbf{x}}_t^i\|^2 - 2\gamma_i \nabla_t^\top (\mathbf{y} - \tilde{\mathbf{x}}_t^i) + \gamma_i^2 \nabla_t^2 \\ \implies \|\tilde{\mathbf{x}}_{t+1}^i\|^2 &\leq \|\tilde{\mathbf{x}}_t^i\|^2 + 2\mathbf{y}^\top (\tilde{\mathbf{x}}_{t+1}^i - \tilde{\mathbf{x}}_t^i) - 2\gamma_i \nabla_t^\top (\mathbf{y} - \tilde{\mathbf{x}}_t^i) + \gamma_i^2 \nabla_t^2 \end{aligned}$$

656 Hence by taking $\mathbf{y} = (1 - \alpha)\mathbf{y}_t \in (1 - \alpha)\mathcal{X}$ and rearranging we get

$$\begin{aligned}
& 2\gamma_i (\ell_t(\tilde{\mathbf{x}}_t^i) - \ell_t((1 - \alpha)\mathbf{y}_t)) \\
&= 2\gamma_i \nabla_t^\top ((1 - \alpha)\mathbf{y}_t - \tilde{\mathbf{x}}_t^i) \\
&\leq \|\tilde{\mathbf{x}}_t^i\|^2 - \|\tilde{\mathbf{x}}_{t+1}^i\|^2 + 2(1 - \alpha)\mathbf{y}_t^\top (\tilde{\mathbf{x}}_{t+1}^i - \tilde{\mathbf{x}}_t^i) + \gamma_i^2 \nabla_t^2
\end{aligned} \tag{43}$$

657 Telescoping with $\tau = 1 \dots t$ we get

$$\begin{aligned}
& \sum_{\tau \in [t]} \ell_\tau(\tilde{\mathbf{x}}_\tau^i) - \sum_{\tau \in [t]} \ell_\tau((1 - \alpha)\mathbf{y}_\tau) \\
&= \frac{1}{2\gamma_i} \|\tilde{\mathbf{x}}_1^i\|^2 + \frac{1 - \alpha}{\gamma_i} \sum_{t \in [T]} \mathbf{y}_t^\top (\tilde{\mathbf{x}}_{t+1}^i - \tilde{\mathbf{x}}_t^i) + \frac{\gamma_i}{2} \sum_{\tau \in [t]} \nabla_\tau^2 \\
&= \frac{1}{2\gamma_i} \|\tilde{\mathbf{x}}_1^i\|^2 + \frac{1 - \alpha}{\gamma_i} \left(\sum_{\tau \in [t-1]} (\mathbf{y}_\tau - \mathbf{y}_{\tau+1})^\top \tilde{\mathbf{x}}_{\tau+1}^i + \mathbf{y}_\tau^\top \tilde{\mathbf{x}}_{\tau+1}^i \right) + \frac{\gamma_i}{2} \sum_{\tau \in [t]} \nabla_\tau^2 \\
&\leq \frac{1}{2\gamma_i} \|\tilde{\mathbf{x}}_1^i\|^2 + \frac{1 - \alpha}{\gamma_i} \sum_{\tau \in [t-1]} (\|\mathbf{y}_\tau - \mathbf{y}_{\tau+1}\| \cdot \|\tilde{\mathbf{x}}_{\tau+1}^i\| + \|\mathbf{y}_\tau\| \cdot \|\tilde{\mathbf{x}}_{\tau+1}^i\|) + \frac{\gamma_i}{2} \sum_{\tau \in [t]} \nabla_\tau^2 \\
&\leq \frac{(1 - \alpha)^2 D^2}{2\gamma_i} + \frac{(1 - \alpha)^2 D}{\gamma_i} (P(\mathbf{y}_{1:T}) + D) + \frac{\gamma_i d^2}{2\rho^2} \left(\bar{F} + K \frac{\bar{F}}{\beta} \bar{G} \right)^2 t
\end{aligned} \tag{44}$$

658 □

659 **Proving (ii):**

660 *Proof.* First, we have for any $t \in [T], i \in [N]$

$$|\ell_t(\tilde{\mathbf{x}}_t^i)| = |\nabla_t^\top (\tilde{\mathbf{x}}_t - \tilde{\mathbf{x}}_t^i)| \leq \|\nabla_t\| \cdot \|\tilde{\mathbf{x}}_t^i - \tilde{\mathbf{x}}_t\| \leq \frac{d}{\rho} \left(\bar{F} + K \frac{\bar{F}}{\beta} \bar{G} \right) \cdot (1 - \alpha) D \tag{45}$$

661 where we recall $D = \sup_{\mathbf{x}, \mathbf{x}' \in \mathcal{X}} \|\mathbf{x} - \mathbf{x}'\|$ is the diameter of $\mathcal{X}$, and both $\tilde{\mathbf{x}}_t^i, \tilde{\mathbf{x}}_t \in (1 - \alpha)\mathcal{X}$.

662 Define $W_t = \sum_{i \in [N]} w_{i,t}$ for all $t \in [T]$, then

$$\begin{aligned}
\log \left(\frac{W_{t+1}}{W_t} \right) &= \log \left(\sum_{i \in [N]} \frac{w_{i,t} \exp(-\epsilon \ell_t(\tilde{\mathbf{x}}_t^i))}{W_t} \right) \\
&= \log \left(\mathbb{E}_{I_t \sim \mathbf{w}_t / W_t} \left[\exp(-\epsilon \ell_t(\tilde{\mathbf{x}}_t^{I_t})) \right] \right) \\
&\stackrel{(a)}{\leq} -\epsilon \mathbb{E}_{I_t \sim \mathbf{w}_t / W_t} \left[\ell_t(\tilde{\mathbf{x}}_t^{I_t}) \right] + \frac{\epsilon^2}{8} \\
&\stackrel{(b)}{=} -\epsilon \ell_t \left(\mathbb{E}_{I_t \sim \mathbf{w}_t / W_t} \left[\tilde{\mathbf{x}}_t^{I_t} \right] \right) + \frac{\epsilon^2}{8} \\
&\stackrel{(c)}{=} -\epsilon \ell_t(\tilde{\mathbf{x}}_t) + \frac{\epsilon^2}{8}
\end{aligned} \tag{46}$$

663 Here (a) follows from Hoeffding's Lemma as described in Lemma B.1 where we take $X = \ell_t(\tilde{\mathbf{x}}_t^{I_t})$,
664 $a =$ and $b =$; (b) follows from the definition that $\ell_t(\tilde{\mathbf{x}}) = \nabla_t^\top (\tilde{\mathbf{x}} - \tilde{\mathbf{x}}_t)$ is a linear function in $\tilde{\mathbf{x}}$; (c)
665 follows from Eq.(5).

666 Hence, telescoping the above we get

$$\log \left(\frac{W_{t+1}}{W_1} \right) \leq -\epsilon \sum_{\tau \in [t]} \ell_\tau(\tilde{\mathbf{x}}_\tau) + \frac{t\epsilon^2}{8} \tag{47}$$

667 On the other hand, we have

$$\begin{aligned}
\log\left(\frac{W_{t+1}}{W_1}\right) &= \log(W_{t+1}) - \log(W_1) \\
&\geq \log\left(\max_{i \in [N]} w_{i,t}\right) - \log(N) \\
&= \max_{i \in [N]} \log(w_{i,t}) - \log(N) \\
&\stackrel{(a)}{=} \max_{i \in [N]} \log\left(w_{i,1} \exp\left(-\epsilon \sum_{\tau \in [t]} \ell_\tau(\tilde{\mathbf{x}}_\tau^i)\right)\right) - \log(N) \\
&= -\epsilon \min_{i \in [N]} \sum_{\tau \in [t]} \ell_\tau(\tilde{\mathbf{x}}_\tau^i) - \log(N)
\end{aligned} \tag{48}$$

668 Hence, combining Eqs.(47) and (48), and dividing both sides by $\epsilon > 0$ we get

$$\begin{aligned}
-\sum_{\tau \in [t]} \ell_\tau(\tilde{\mathbf{x}}_\tau) + \frac{t\epsilon}{8} &\geq -\min_{i \in [N]} \sum_{\tau \in [t]} \ell_\tau(\tilde{\mathbf{x}}_\tau^i) - \frac{\log(N)}{\epsilon} \\
\Rightarrow \sum_{\tau \in [t]} \ell_\tau(\tilde{\mathbf{x}}_\tau) - \min_{i \in [N]} \sum_{\tau \in [t]} \ell_\tau(\tilde{\mathbf{x}}_\tau^i) &\leq \frac{t\epsilon}{8} + \frac{\log(N)}{\epsilon} \\
\Rightarrow \sum_{\tau \in [t]} \ell_\tau(\tilde{\mathbf{x}}_\tau) - \sum_{\tau \in [t]} \ell_\tau(\tilde{\mathbf{x}}_\tau^i) &\leq \frac{t\epsilon}{8} + \frac{\log(N)}{\epsilon}, \quad \forall i \in [N]
\end{aligned} \tag{49}$$

669

□

670 B Supplementary lemmas

671 **Lemma B.1** (Hoeffding's lemma). *Let X be some random variable such that $a \leq X \leq b$ almost*
672 *surely for some $a, b \in \mathbb{R}$. Then for any $\epsilon \in \mathbb{R}$, we have $\mathbb{E}[\exp(-\epsilon X)] \leq \exp\left(-\epsilon \mathbb{E}[X] + \frac{\epsilon^2(b-a)^2}{8}\right)$.*

673 **Lemma B.2** ([25] Lemma 2.1). *Let $h : \mathcal{X} \rightarrow \mathbb{R}$ be some convex function (not necessarily differen-*
674 *tiable). Then for any $\mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^d$ and $\delta > 0$ we have*

$$\nabla_{\mathbf{x}} \mathbb{E}_{\mathbf{v} \sim U(\mathbb{B})} [h(\mathbf{x} + \delta \mathbf{v})] = \frac{d}{\delta} \cdot \mathbb{E}_{\mathbf{u} \sim U(\mathbb{S})} [h(\mathbf{x} + \delta \mathbf{u}) \cdot \mathbf{u}] \tag{50}$$